The Area Problem

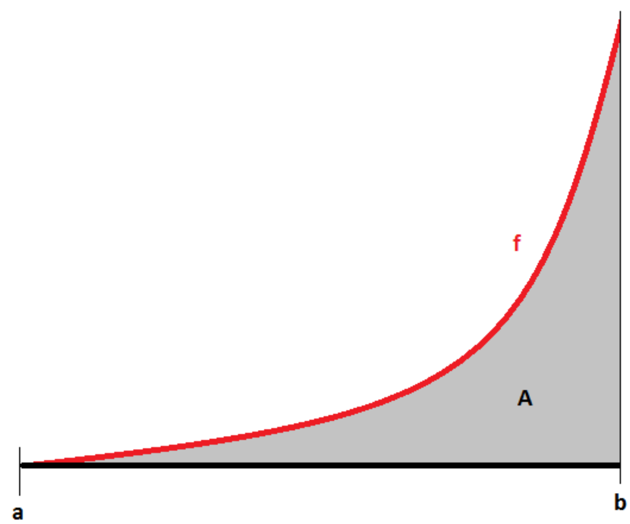
Peyam Ryan Tabrizian

Tuesday, November 9th, 2010

1 The Area Problem

In this last part of the course, we are interested in calculating areas, and in particular finding a fast way of doing so! (notice the similarity with what we did for derivatives). For example, given the graph of a function f, how can we calculate the area of region A in the picture below?

1A/Handouts/Area.png



(In the problem in the next section, we will have $f(x) = x^3$, $a = 0$, $b = 1$)

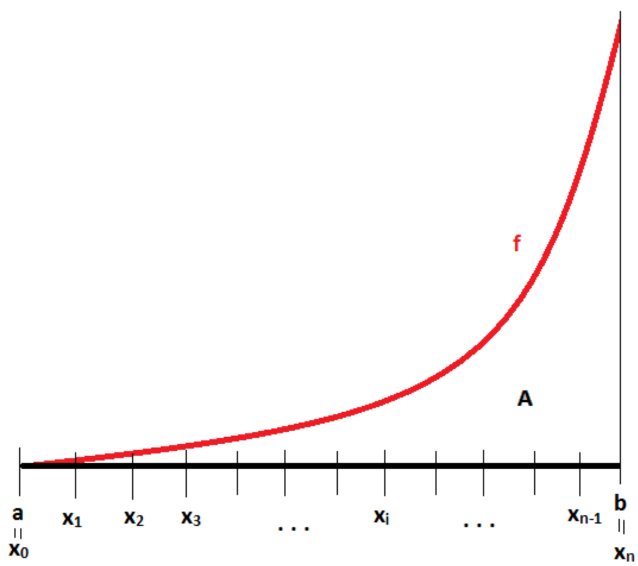
Now this seems like a pretty hard task, but remember the general principle mentioned several times this semester:

General Principle: Whenever you don't know how to calculate something, first calculate something you know, and then take a limit!

Here, we'll see this principle in action! What we know is to calculate areas of rectangles (the formula is easy, it's just length $\times$ width!), so our strategy will be to compute areas of rectangles, and then take some sort of a limit.

To do this, we first divide the interval $[a, b]$ into n even pieces. Think of it just like cutting a cake, except that your cake is an interval! So we get something like in the following picture:

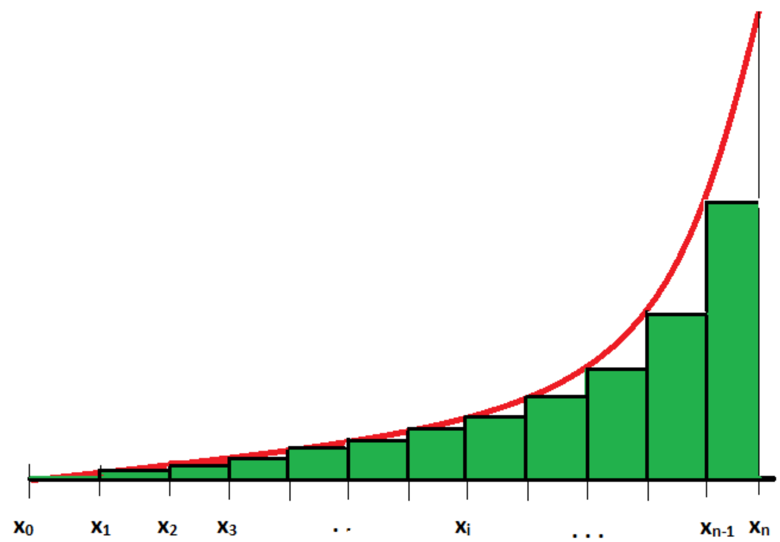
1A/Handouts/Partition.png



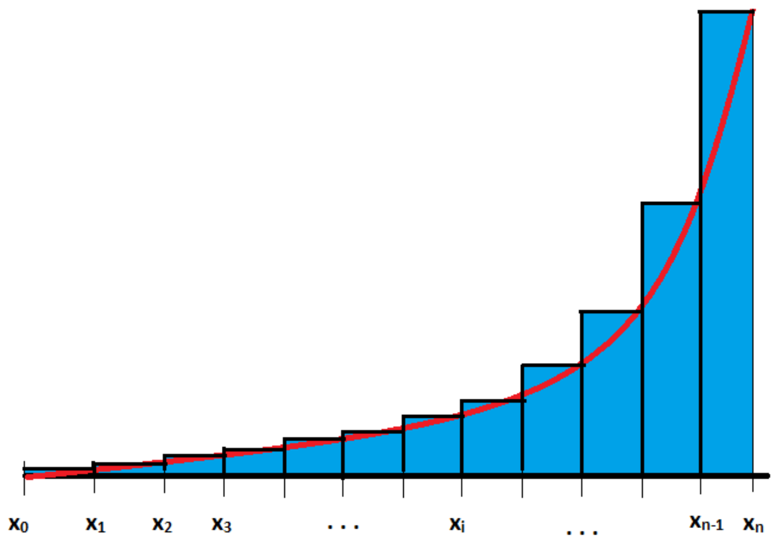
And notice that the every piece has length $\frac{b-a}{n}$ (because we divided $[a, b]$ into n even pieces). Now notice that in the picture above, there are some weird symbols like x_0, x_1 or even x_i, x_n. They are your best friends, so don't be afraid of them! They just say that we divided the interval $[a, b]$ into n mini-intervals, namely $[x_0, x_1]$, $[x_1, x_2]$, $[x_2, x_3]$, and so on, until $[x_{n-1}, x_n]$. And we also have $x_0 = a$, $x_n = b$. If you're driving a car from a to b, think of the x_i as pitstops. Finally, don't be scared about the term x_i (I was when I was a freshman :)). It just means that the *general term* is x_i. This is similar to when we say $f(x) = x^2$, imagine how tedious it would be, instead of writing x^2, to say that we have a function f such that $f(1) = 1$, $f(2) = 4$, $f(3) = 9$, etc. It is easier just to say $f(x) = x^2$. Here it is the same thing! Instead of writing $x_0, x_1, x_2, \cdots$, we just write x_i. It's the same idea, and nothing more!

Now, once we divided $[a, b]$ up into n mini-intervals $[x_i, x_{i+1}]$, we need to cook up rectangles which approximate the area A. There are many, many ways of doing that, but two of them are very important for this course:

1A/Handouts/Ln.png



1A/Handouts/Rn.png

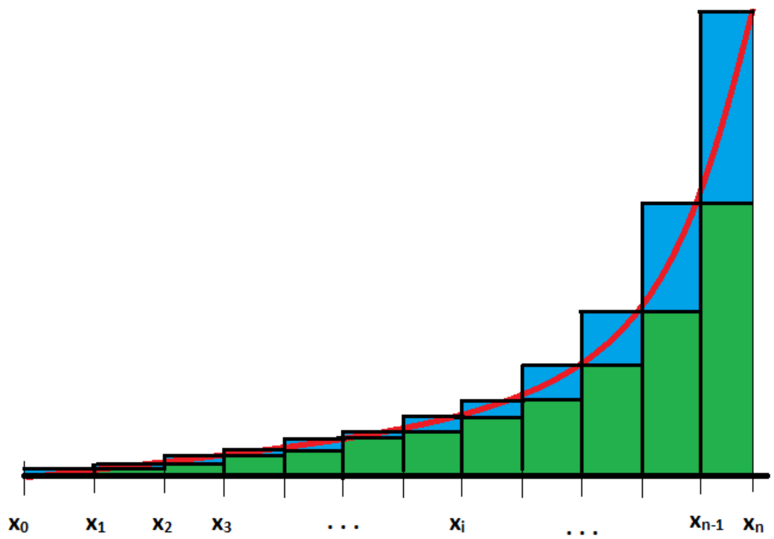


The first one is the left-hand-sum L_n. On each interval $[x_i, x_{i+1}]$, you consider the rectangle of height $f(x_i)$. For example, your first rectangle will have height $f(x_0)$, your second rectangle height $f(x_1)$, and your n^{th} rectangle will have height $f(x_{n-1})$. Basically, you'll start at x_0 and end at x_{n+1}.

The second one is similar, and called the right-hand-sum R_n. On each interval $[x_i, x_{i+1}]$, you consider the rectangle of height $f(x_{i+1})$. For example, your first rectangle will have height $f(x_1)$, your second rectangle height $f(x_2)$, and your n^{th} rectangle will have height $f(x_n)$. Basically, you'll start at x_1 and end at x_n.

The following picture shows that the two approaches above are not exactly the same (the two pictures above might be misleading in this sense):

1A/Handouts/Comparison.png



Again, to summarize, the Left-Hand-Sum essentially starts at x_0 and ends at x_{n-1}, and the Right-Hand-Sum starts at x_1 and ends at x_n. You can think of it in the following way, if you want: Imagine you are taking a class with $n+1$ homework assignments, labeled homework 0 up to homework n and only n of them count. There are many ways of turning in your homework, but the two basic strategies are: Be productive, and start with homework 0, and stop at homework $n-1$ because you only have to turn in n assignments. This is the same as the left-hand-sum. Or take it easy at first, and start with homework 1, and work all the way through homework n. This is the right-hand-sum.

Now, for each approach (left-hand-sum and right-hand-sum), we'd like to compute the sum of the areas of each rectangle (hence the name left-hand-**sum**). Again, notice that in this case, each rectangle has width $\frac{b-a}{n}$ (notice that this doesn't depend on the rectangle used! On Wednesday, you'll do a different sum where the width depends on the rectangle). And for the length, for the left-hand-sum, the i^{th} rectangle has height $f(x_i)$ (i is between 0 and $n-1$). And for the right-hand-sum, the i^{th} rectangle has height $f(x_{i+1})$ (again, i is between 0 and $n-1$). Finally, using the formula Area = Width $\times$ Height, we can explicitly compute L_n an R_n, namely:

$$\begin{aligned} L_n &= \left(\frac{b-a}{n}\right) f(x_0) + \left(\frac{b-a}{n}\right) f(x_1) + \left(\frac{b-a}{n}\right) f(x_2) + \cdots + \left(\frac{b-a}{n}\right) f(x_{n-1}) \\ &= \left(\frac{b-a}{n}\right) \left(f(x_0) + f(x_1) + \cdots + f(x_{n-1})\right) \end{aligned}$$

$$\begin{aligned} R_n &= \left(\frac{b-a}{n}\right) f(x_1) + \left(\frac{b-a}{n}\right) f(x_2) + \left(\frac{b-a}{n}\right) f(x_3) + \cdots + \left(\frac{b-a}{n}\right) f(x_n) \\ &= \left(\frac{b-a}{n}\right) \left(f(x_1) + f(x_2) + \cdots + f(x_{n-1}) + f(x_n)\right) \end{aligned}$$

Awesome!!! We found a good formula for L_n and R_n! The only thing we still need to do is to take limits, namely:

$$A = \lim_{n\to\infty} L_n = \lim_{n\to\infty} R_n$$

It is **NOT OBVIOUS** that both approaches give the same limit, but in most cases it's true, and we'll see very soon for which cases the above holds.

We summarize our approach into a method for calculating the area under the graph of a function f from a to b:

Method:

- Calculate $R_n = \left(\frac{b-a}{n}\right) \left(f(x_1) + f(x_2) + \cdots + f(x_{n-1}) + f(x_n)\right)$

- Take $\lim_{n\to\infty} R_n$

Three remarks are in order. First, **DO NOT** memorize this formula! Understand what is going on, and it will be very easy for you to re-derive it! Second, using the right-hand-sum is **enough** for this problem, you don't have to calculate both R_n and L_n. But sometimes, you might be asked: Compute this area using Left-Hand-Sums! Finally, notice that one thing that simplified this formula was that the width of the rectangles were all the same! Prof. Christ will talk about the case where the widths are not the same on Wednesday!

2 Sample Problems

After this **looooooooooong** discussion, we can finally move on to sample problems!

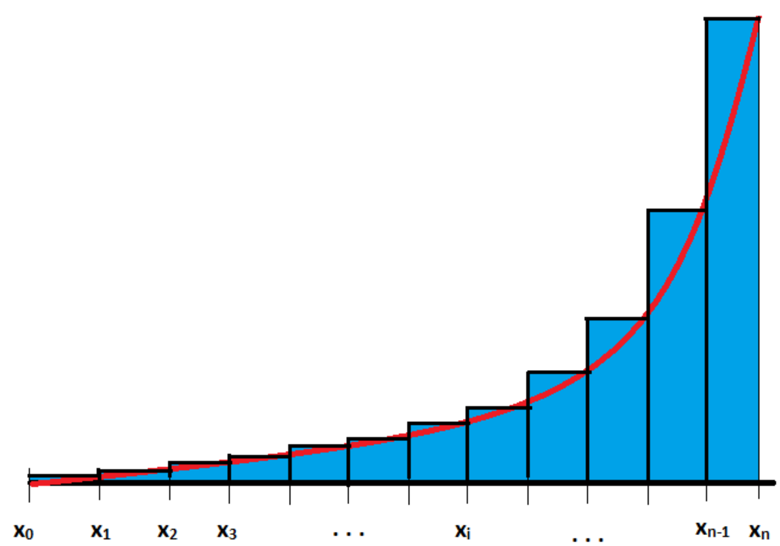
2.1 Sample Problem 1: Find the area under the graph of $f(x) = x^3$ from $x = 0$ to $x = 1$

.

All the work we did in the previous part greatly simplifies our task! Notice that here $f(x) = x^3$, $a = 0$, and $b = 1$. Now, as above, we divide $[0, 1]$ into n even mini-intervals, each of which has length $\frac{1}{n}$.

Then, the next step is to find R_n, and we do this by first drawing a picture exactly like above (included here for your convenience):

1A/Handouts/Rn.png



The only thing you would change is to indicate that $x_0 = 0$ and $x_n = 1$.

Now notice that we can explicitly calculate x_i, taking into account that the x_i are evenly spaced in the interval $[0, 1]$.

So we get: $x_0 = 0$, $x_1 = x_0 + \frac{1}{n} = \frac{1}{n}$, $x_2 = x_1 + \frac{1}{n} = \frac{1}{n} + \frac{1}{n} = \frac{2}{n}$, $x_3 = x_2 + \frac{1}{n} = \frac{2}{n} + \frac{1}{n} = \frac{3}{n}$. By now you should have noticed the pattern! The general term is $x_i = \frac{i}{n}$, and in particular $x_n = \frac{n}{n} = 1$.

Now taking into account that the width of the rectangles are $\frac{1}{n}$, we can calculate R_n:

$$
\begin{aligned}
R_n &= \left(\frac{1}{n}\right)\left(f(x_1) + f(x_2) + \cdots + f(x_{n-1}) + f(x_n)\right) \\
&= \left(\frac{1}{n}\right)\left((x_1)^3 + (x_2)^3 + \cdots + (x_i)^3 + \cdots + (x_{n-1})^3 + (x_n)^3\right) \\
&= \left(\frac{1}{n}\right)\left(\left(\frac{1}{n}\right)^3 + \left(\frac{2}{n}\right)^3 + \cdots + \left(\frac{i}{n}\right)^3 + \left(\frac{n-1}{n}\right)^3 + \left(\frac{n}{n}\right)^3\right) \\
&= \left(\frac{1}{n}\right)^4 \left(1^3 + 2^3 + \cdots + i^3 + \cdots (n-1)^3 + n^3\right) \\
&= \left(\frac{1}{n}\right)^4 \left(\frac{n^2(n+1)^2}{4}\right) \\
R_n &= \frac{(n+1)^2}{4n^2}
\end{aligned}
$$

Finally, we take $\lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{(n+1)^2}{4n^2} = \frac{1}{4}$.

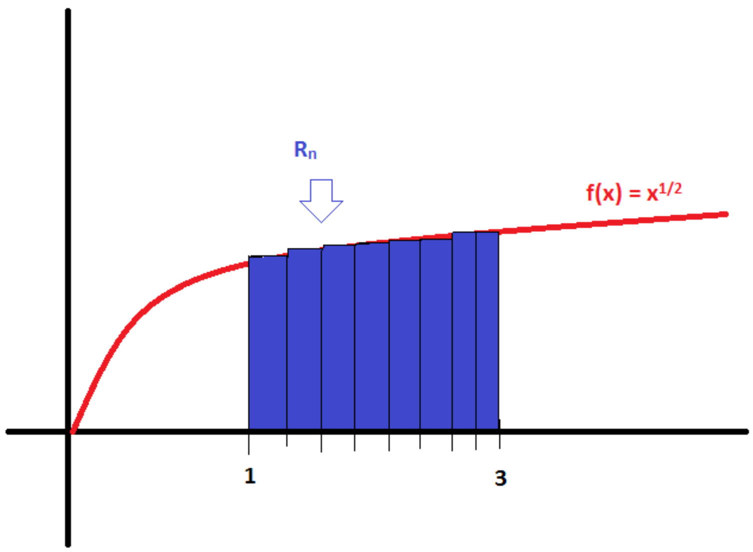
So our desired area is $\boxed{\text{A} = \frac{1}{4}}$.

Note: Notice how we needed the formula for the sum of cubes here! Usually, this formula will be given to you!

2.2 Find a formula for the area under the graph of $f(x) = \sqrt{x}$ from $x = 1$ to $x = 3$, but do not evaluate it!

This problem basically says: Find R_n but do not compute $\lim_{n\to\infty} R_n$! The picture is basically the same as the picture above, except that now we are dealing with the square root function, and furthermore we are considering x between 1 and 3.

1A/Handouts/Square root.png



The strategy is again the same: Divide $[1,3]$ into n even pieces. Each has length $\frac{3-1}{n} = \frac{2}{n}$.

Again, we can compute x_i explicitly: $x_0 = 1$, $x_1 = 1 + \frac{2}{n}$, $x_2 = 1 + \frac{4}{n}$, $x_3 = 1 + \frac{6}{n}$, and in general, $x_i = 1 + i\left(\frac{2}{n}\right)$ (because the length here is $\frac{2}{n}$).

So we get:

$$
\begin{aligned}
R_n &= \left(\frac{2}{n}\right)\left(f(x_1) + f(x_2) + \cdots + f(x_{n-1}) + f(x_n)\right) \\
&= \left(\frac{2}{n}\right)\left(\sqrt{x_1} + \sqrt{x_2} + \cdots + \sqrt{x_i} + \cdots + \sqrt{x_{n-1}} + \sqrt{x_n}\right) \\
&= \left(\frac{2}{n}\right)\left(\sqrt{1+\frac{2}{n}} + \sqrt{1+\frac{4}{n}} + \cdots + \sqrt{1+i\cdot\frac{2}{n}} + \cdots + \sqrt{1+\frac{2(n-1)}{n}} + \sqrt{3}\right)
\end{aligned}
$$

And finally $A = \lim_{n\to\infty} R_n$, i.e. you take the above formula and let n go to ∞

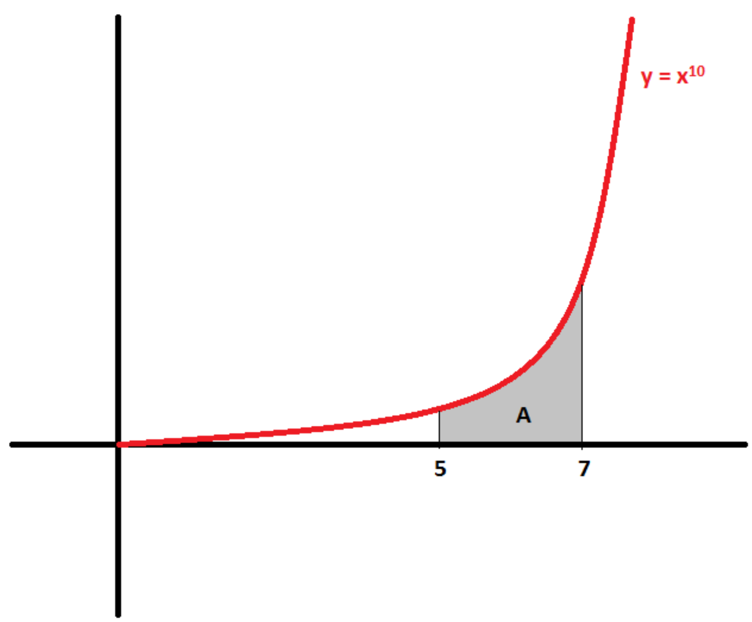
2.3 Determine a region whose area is equal to the gien limit:

The limit here is:

$$\lim_{n\to\infty}\left(\frac{\mathbf{2}}{\mathbf{n}}\right)\left(\left(\mathbf{5}+\frac{2}{n}\right)^{\mathbf{10}}+\left(5+\frac{4}{n}\right)^{10}+\cdots+\left(5+\frac{2i}{n}\right)^{10}+\cdots+(\mathbf{7})^{10}\right)$$

You should **immediately** recognize this as $\lim_{n\to\infty} R_n$. The important parts to focus on are in bold. First of all, we learn that the width of the rectangles in this sum is $\frac{2}{n}$, and that the function in question is $f(x) = x^{10}$, because of the 'power-of-10' term ($f(x) = (x+5)^1 0$ would also work, but your endpoints are different). Also, the first term x_1 is $5 + \frac{2}{n}$, so our guess is that $a = 5$ and finally, the x_n term is 7, so we get $b = 7$.

Hence, the above limit represents the area of the region A under the function $f(x) = x^{10}$ from $a = 5$ to $b = 7$. You could draw a picture if you want to. It would look somehow like this:



1A/Handouts/x^{1}0*.png*